leigh number of 10^6. The overall thermal conductivity value varies between a maximum of 4.23 at a Reynolds number of 416 and a minimum value of 4.19 at a Reynolds number of 1120. From the preceding studies, it can be concluded that for a given Rayleigh number in the range of 10^3-10^6, the overall equivalent thermal conductivity is almost constant with respect to the rotational Reynolds number in the range of 0-10^3, although the streamlines, isotherms, and local equivalent thermal conductivity exhibits very different features.

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Two-Dimensional Effects in a Triangular Convecting Fin

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Introduction

THE traditional approach to the analysis and design of fins is based on the assumption of one-dimensional conduction in the fin material. In recent years, however, several workers have studied the effect of two-dimensional conduction on the performance of fins. For example, Lau and Tan[1] used the method of separation of variables to develop exact solutions for two-dimensional conduction in straight and annular fins of rectangular profile. A similar approach for a cylindrical fin was used by Irey.[2] These exact solutions show that the one-dimensional assumption is valid only if the Biot number (Bi) based on half thickness of the fin is much less then one, otherwise the errors can be as high as 60% for $Bi = 10$. The validity criterion of $Bi \ll 1$ was established analytically by Levitsky[3] who showed that the exact solution for three-dimensional conduction in an infinitely long rectangular fin does indeed reduce to the standard one-dimensional solution if $Bi \ll 1$.

Since the two-dimensional triangular fin does not admit an exact analytical solution, the problem has been studied by Sfeir[4] and Burmeister[5] using an approximate technique. Both authors used a heat balance integral approach to reduce the two-dimensional heat conduction equation to an ordinary differential equation. While Sfeir solved the equation numerically, Burmeister obtained the solution in terms of hypergeometric functions. Since numerical solutions of the problem are not available, it is not possible to assess the accuracy of these approximate results. In any case, the results presented in these two papers do not provide a quantitative picture of how the Biot number and length-to-base thickness ratio affect the heat transfer rate and the magnitude of the errors introduced due to the one-dimensional assumption.

The purpose of this work is twofold. The first purpose is to obtain a finite element solution for the two-dimensional triangular fin and present the heat transfer data for a wide range of Biot number and length-to-base thickness ratio so that the information can be used for prediction as well as design purposes. The second purpose is to report additional data for the rectangular fin to supplement the results of Lau and Tan.[1]

Analysis

Consider a triangular fin of length L and base thickness w attached to a primary surface at temperature T_b (shown as inset in Fig. 2). The fin is convecting heat from both its sloping faces to an environment at temperature T_∞, the heat transfer coefficient being h. Because of thermal symmetry, we analyze one-half of the fin. For two-dimensional conduction, the governing equation and boundary conditions in dimensionless form are

$$\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} = 0 \tag{1}$$

$$\theta(0,Y) = 1, \quad \frac{\partial\theta}{\partial Y}(X,0) = 0, \quad \nabla\theta\cdot\vec{n} = -Bi\,\theta \text{ on sloping surface} \tag{2}$$

where $\theta = (T - T_\infty)/(T_b - T_\infty)$, $X = \alpha x/L$, $Y = y/(w/2)$, $Bi = hw/2k$, $\alpha = L/(w/2)$, and $\vec{n}$ is the outward normal vector. In Eq. (2), the first condition represents the constant base (root) temperature. The second condition is the result of thermal symmetry (symmetrical temperature distribution about $y = 0$). The last condition corresponds to convective dissipation from the sloping surface to the environment.

The total heat transfer, q_2 (subscript 2 denotes the two-dimensional solution) per unit depth of the fin can be expressed in dimensionless form as

$$Q_2 = \frac{q_2}{k\,(T_b - T_\infty)} = -\frac{2}{\alpha}\int_0^1 \left.\frac{\partial\theta}{\partial X}\right|_{X=0} \mathrm{d}Y \tag{3}$$

Equations (1) and (2) were solved using a finite element approach. The solution was subsequently used to evaluate Q_2

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from Eq. (3). The triangular domain was divided to give a total of 225 nodal points and 441 elements, the nodes being closely spaced near the base to accommodate the sharp temperature gradients near the base (root). A brief discussion of the grid refinement study appears in the following section.

Results and Discussion

The finite element program was first used to compute the heat transfer for the two-dimensional rectangular fin with constant base temperature and a uniform heat transfer coefficient for all convecting surface including the fin tip. Heat transfer results were obtained for $\alpha = 2L/w = 0.1, 0.5, 1.0, 5.0, 10, 50, 100$, and 500 for each of the following Biot number: $Bi = 0.01, 0.10, 0.25, 0.50, 1, 2, 4, 6, 8$, and 10. These results are shown in Fig. 1. The exact analytical solution of Lau and Tan[1] for $Bi = 0.01$, and 0.1, and the finite difference solution of Sfeir[4] for $Bi = 1$ are shown by dashed lines. These results are hardly distinguishable from the present finite element solutions. For $Bi \leq 0.5$, the heat transfer rate Q_2 increases as the length-to-base thickness ratio α increases, and eventually approaches an asymptotic value. For $\alpha > 50$, Q_2 is essentially a function of Bi alone, increasing as Bi increases. For $Bi \geq 1$, the effect of α on Q_2 is opposite to that for $Bi \leq 0.5$. As α increases, Q_2 decreases, reaching asymptotic values for $\alpha > 10$. Therefore, the conclusion is that $Q_2 = f(\alpha, Bi)$ for $0.1 \leq \alpha \leq 10$ but for $\alpha > 10$, $Q_2 \simeq f(Bi)$. The present results for the two-dimensional rectangular fin were compared with the corresponding one-dimensional solutions to assess the error. However, space limitations preclude presentation of quantitative results, and only the conclusions are highlighted here. The general conclusion is that the one-dimensional solution overpredicts the heat transfer rate. For each Bi, the error increases as α increases and levels off for $\alpha > 10$. For $Bi \leq 0.1$, the error is under one percent and for $Bi \leq 1$, the error is under 10%. However, for $Bi = 10$, the error reaches a value of 60% for $\alpha \geq 1$. These confirm the earlier conclusions of Lau and Tan.[1] Since Fig. 1 provides data for a number of values of Bi in the range from 0.01 to 10, it is more comprehensive than the heat transfer results of Lau and Tan,[1] which pertain to $Bi = 0.01, 0.05$, and 0.1. Thus, Fig. 1 can serve as a useful chart for predicting the two-dimensional performance of a rectangular fin under a wide range of operating conditions.

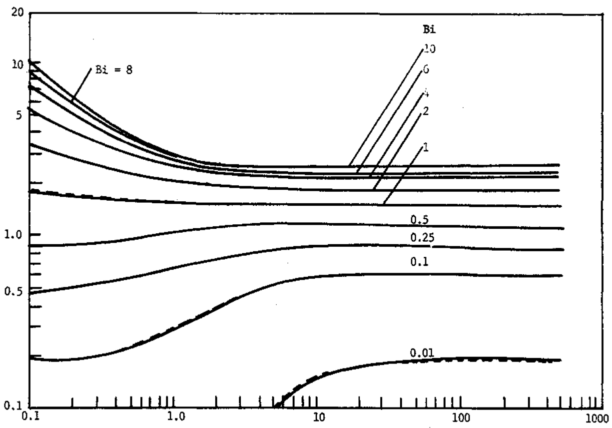

Fig. 1 Two-dimensional heat transfer in a straight rectangular fin: —— present solution, - - - - finite difference and analytical solutions.

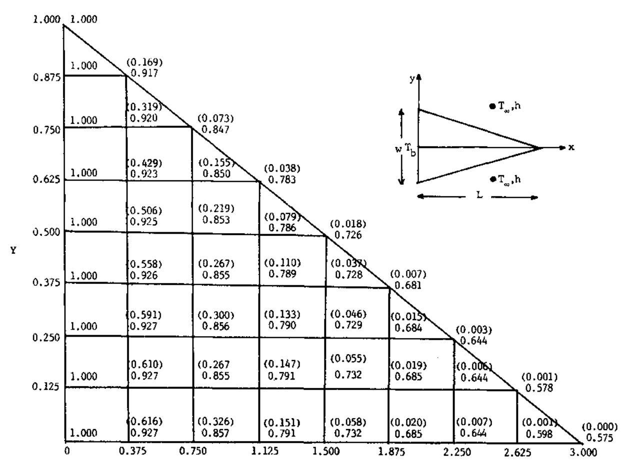

Fig. 2 Temperature distributions in two-dimensional triangular fin for $\alpha = 3$ and $Bi = 0.1$ and 10; results for $Bi = 10$ appear in parenthesis.

Next, we consider the results for a triangular fin. A sample temperature distribution is shown in Fig. 2 for $\alpha = 3$ (fin length is 1.5 times its base thickness). Two sets of results are shown, one for $Bi = 0.1$ and the other for $Bi = 10$. The results for $Bi = 0.1$ show that the conduction is essentially one-dimensional, the temperature variations along y direction being very small. However, when $Bi = 10$ (shown in parenthesis), the two-dimensional effect becomes quite prominent and the temperature varies significantly along the y direction. However, any general conclusion regarding the two-dimensional effect is deferred to a latter section.

A sample of results of grid refinement study is shown in Table 1. The temperature values along the centerline ($y = 0$) of the fin are given for 66, 100, and 225 nodes. The change in grid size from 100 nodes to 225 produced a change only in the fourth decimal place, which indicates satisfactory convergence of the numerical procedure.

Fig. 3 depicts the heat transfer results for a two-dimensional triangular fin for the same values of α and Bi as were used in Fig. 1. The curves exhibit the same kind of general behavior. However, it must be noted that for a given α and Bi, the heat transfer from a triangular fin is higher than its rectangular counterpart. Another difference is that the curves for $Bi = 0.01, 0.1, 0.25, 0.5, 0.75$, and 1.0 for the triangular fin exhibit maxima which were absent in the case of the rectangular fin.

To assess the error due to the one-dimensional assumption, the two-dimensional heat transfer Q_2 was compared to the one-dimensional heat transfer Q_1. The error was calculated as $100\,(Q_1 - Q_2)\,/\,Q_2$. The standard one-dimensional solution[6] for Q_1 expressed in terms of the present nomenclature is

$$Q_1 = \frac{q_1}{k(T_b - T_\alpha)} = 2\sqrt{Bi}\,\frac{I_1(2\alpha\sqrt{Bi})}{I_0(2\alpha\sqrt{Bi})} \tag{4}$$

where I_0 and I_1 are the modified Bessel functions of first kind. Since the generally available tables of $I_0(x)$ and $I_1(x)$ do not

Table 1 Temperature distribution in a triangular fin at $y = 0$ for $Bi = 0.1$, $\alpha = 3$: Results of grid refinement study

X	66 Nodes	100 Nodes	225 Nodes
0	1.0000	1.0000	1.0000
0.375	0.9268	0.9271	0.9273
0.750	0.8746	0.8750	0.8752
1.125	0.7909	0.7912	0.7914
1.500	0.7316	0.7320	0.7323
1.875	0.6842	0.6846	0.6849
2.250	0.6437	0.6440	0.6442
2.625	0.5970	0.5977	0.5979
3.000	0.5742	0.5749	0.5750

Table 2 Combination of α and Bi for which one-dimensional assumption gives negligible error

α	1	2	3	4	5	10	50	100
Bi	8.0	3.0	1.5	0.80	0.65	0.27	0.01	0.01

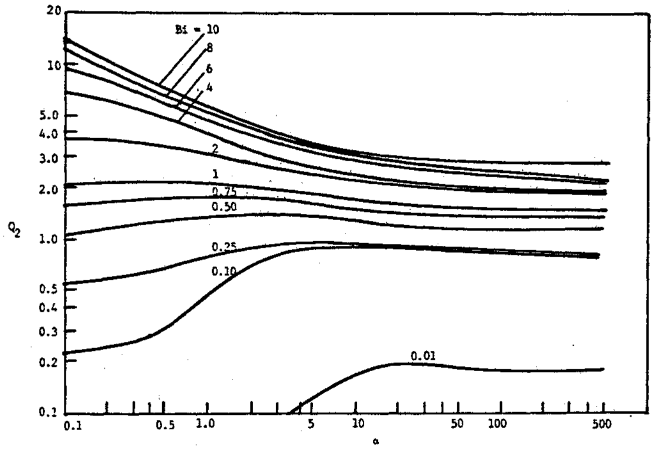

Fig. 3 Two-dimensional heat transfer rate in a straight triangular fin.

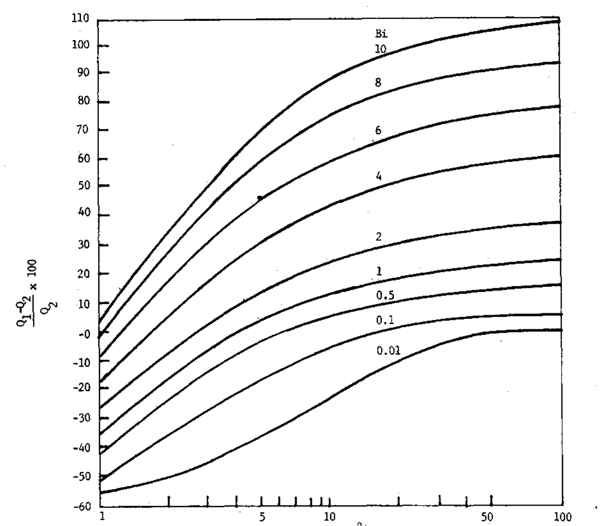

Fig. 4 Error in heat transfer rate in a triangular fin.

go beyond $x = 10$, I_0, and I_1 for certain combinations of α and Bi had to be evaluated from their series expansions. The percent error in heat transfer rate is plotted against α for various values of Bi in Fig. 4. It shows that the error can be positive or negative depending on the values of α and Bi. The error ranges from -55% to $+110\%$. It is interesting to note that the criterion of $Bi << 1$ generally accepted for the validity of one-dimensional assumption in a rectangular fin fails in the case of the triangular fin. For example, if the fin is thick and short, say $\alpha = 1$, the error is about -55% even when $Bi = 0.01$. On the other hand, the same fin can be treated as one-dimensional (with an error $+7\%$) at $Bi = 10$. Thus, it is not possible to establish a validity criterion for a triangular fin in terms of Bi number alone. It is the combination of α and Bi that dictate whether or not the one-dimensional assumption is valid. Table 2 shows the combinations for which the one-dimensional assumption will result in negligible error.

When using the results of this investigation, it is important to recognize the limitations of the analysis. One limitation is the assumption of a constant heat transfer coefficient, h. Studies by Stachiewicz,[7] Unal,[8] and Look[9] have shown that h can vary significantly along the convecting surface. Another limitation is that the base temperature T_b is spatially uniform. It is known from the works of Sparrow and Hennecke,[10] Klett and McCulloch,[11] and Look[12] that the fin base temperature is depressed and becomes nonuniform as a result of the thermal interaction between the fin and the primary surface to which it is attached. In view of these and other limitations that characterize many fin studies, the results of the present investigation, like any other investigation, must be used with caution.

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Entrance Heat Transfer in Isosceles and Right Triangular Ducts

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Introduction

HEAT transfer from isosceles and right triangular ducts to an incompressible fluid with constant physical properties is presented. The flow is laminar and hydrodynamically fully developed; however, it develops thermally with the walls maintained at a constant temperature. Axial conduction, viscous dissipation, flow work, and thermal energy generation within the fluid are neglected.

Heat transfer in noncircular ducts is of considerable importance in the design of compact heat exchangers. A Galerkin-based integral method is used to carry out the computations. A generalized procedure that results in a closed-form solution for noncircular ducts was reported earlier.[1] The data presented here are comprehensive and cover a wider range than those in Ref. 1.

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